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# A class of similarity solutions for the nonlinear thermal conduction problem

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**Abstract.** A class of self-similar solutions are rederived for the nonlinear thermal conduction problem. Variational methods applied to the heat flux provide an expression valid over a broad range of physical parameters. The results are particularly useful for the understanding of heat flux in the plasmas of inertial confinement fusion. They describe both rapid laser and radiant heating of matter. The developments are shown to correspond well to several classical examples.

## 1. Introduction

Nonlinear conduction of heat in the early phases of rapid energy input to matter is a classical problem in the field of high-temperature hydrodynamics. Its application to the physics of inertial confinement fusion has prompted significant renewed interest. Similarity solutions were examined by Marshak (1958) and more extensively by Zeldovich and Raizer (1967). Marshak found approximate solutions for a thermal conduction wave driven (in planar geometry) by an exponentially rising temperature at the material boundary, whereas Zeldovich and Raizer described analytic solutions for a thermal wave driven by an instantaneous deposition of energy at the material boundary.

The mathematics of these similarity solutions has been extensively developed in the literature. Gilding and Peletier (1976) provide a comprehensive discussion of the topic as well as a survey of the research that contributed to its development. Grundy (1979) formulates a phase plane analysis of the similarity solution for the conduction equation. This reveals a complete picture of its existence and uniqueness properties. It also permits an examination of the behaviour of the solutions in the phase plane.

In this paper a full development of the details is restated for a particular form of the similarity substitution for the nonlinear thermal conduction problem. At the same time the special cases of Marshak and Zeldovich, mentioned above, are tied together in the broad class of self-similar solutions. Then close approximate solutions are obtained for the intermediate cases in order to illustrate the scaling of the thermal wave parameters. Well known variational methods (Becker 1964) are employed to generate the family of approximate solutions. The combination of these mathematical techniques provides a clear picture of the physics of thermal conduction in inertial fusion plasmas.

## 2. Similarity variables

Assuming constant density, the standard form of the nonlinear thermal conduction equation is given by

$$\partial T/\partial t = A\nabla \cdot T^n \nabla T \tag{1}$$

where $T$ is the material temperature, $A = \kappa_0/\rho c_v$ is a constant, and $\kappa \sim \kappa_0 T^n$ is the thermal conductivity. The notation is that of Zeldovich and Raizer (1967) where $\rho$ is the density and $c_v$ is the specific heat at constant volume. The energy flux

$$\boldsymbol{S} = -\rho c_v A T^n \nabla T \tag{2}$$

can be either electron thermal flux, where $n = \frac{5}{2}$, or radiation flux, where $n \approx 4$–$5$.

Self-similar solutions to equation (1) in planar geometry can be most easily found by assuming a product solution of the form

$$T(x, t) = T_0 g(t) h(\xi) \tag{3}$$

where the similarity variable

$$\xi = x/(Z(t)L_0). \tag{4}$$

$L_0$ is an initial length scale to be defined below, and $g(0) = h(0) = 1$. Notice that $g(t)$ describes the temperature dependence at the boundary (initial temperature $T_0$). Using (3) and (4) in (1), we obtain

$$(Z^2/g^{n+1})(\dot{g}h - gh'\xi\dot{Z}/Z) = (h^n h')'(AT_0^n/L_0^2) \tag{5}$$

where the dots are time derivatives and the prime is a derivative with respect to $\xi$. If $\dot{Z} \neq 0$, we can rewrite (5) as

$$(Z\dot{Z}/g^n)(\dot{g}Z/g\dot{Z} - \xi h'/h) = (h^n h')' h^{-1}(AT_0^n/L_0^2). \tag{6}$$

For (6) to be satisfied, we must require

$$\dot{g}Z/g\dot{Z} \equiv \beta \tag{7}$$

and

$$Z\dot{Z}/g^n \equiv \nu \tag{8}$$

for a pair of constants $\beta$ and $\nu$. Choose the solution $g = Z^\beta$ for equation (7). Thus, $Z(0) = 1$ since $g(0) = 1$. Substitute this result in (8) to obtain

$$Z\dot{Z} = \nu Z^{\beta n}. \tag{9}$$

Then, for $\beta n \neq 2$,

$$Z = [1 + (2 - \beta n)\nu t]^{1/(2-\beta n)} \tag{10}$$

and, for $\beta n = 2$,

$$Z = e^{\nu t} \tag{11}$$

where we have satisfied the boundary condition $Z(0) = 1$.

It is important to notice that the parameters $\nu$ and $\beta$ are restricted by the nature of the physical problem. From equation (4) the velocity of the heat front located at $\xi = \xi_0$ is given by

$$\dot{x}_f = \xi_0 L_0 \dot{Z}. \tag{12}$$

Thus both equations (10) and (11) indicate that we must have $\nu > 0$ for the heat wave to be forward-going. The domain of $\beta$ is restricted to $\beta \leqslant 2/n$ in order to avoid infinite velocities.

With the above transformations and with the initial scale length defined by $L_0^2 = AT_0^n\nu^{-1}$, we are left with the differential equation in the similarity variable, $\xi$,

$$(h^n h')' = \beta h - \xi h'. \tag{13}$$

The boundary conditions are $h(0) = 1$, and the requirement that the flux $S \sim h^n h'$ goes to zero at some point $\xi_0 > 0$ inside the material.

## 3. Variational solutions

Although standard numerical techniques can be readily applied to the integration of equation (13), it is instructive to obtain approximate analytic solutions over a large parameter domain. To do this, it is helpful to use the following transformations. We let $h = \varphi^{1/n}$ and $u = \xi/\xi_0$, such that $h(\xi_0) = \varphi^{1/n}(1) = 0$. Equation (13) becomes

$$n\varphi\varphi'' + (\varphi')^2 = K(\beta n\varphi - u\varphi') \tag{14}$$

where $K = \xi_0^2 n$, and the prime is a derivative with respect to $u$.

The equation is now cast in the form of a nonlinear eigenvalue problem, where we must find the positive value of $K$ such that $\varphi$ is monotone decreasing on the interval $0 \leqslant u \leqslant 1$ and goes to zero at $u = 1$. To determine approximate (and useful) analytic solutions to (14), we have employed a least squares variational technique (Becker 1964) and generated an expression for the corresponding eigenvalue $K$. A schematic description of our approach to the problem follows.

We choose a trial function $\varphi_T$ given by

$$\varphi_T = 1 - au + (a-1)u^2. \tag{15}$$

This function is selected for several reasons. Besides satisfying the boundary conditions, it is an exact solution in the limit $\beta = -1$ (Zeldovich's solution) and is approximately correct for $\beta = 2/n$ (Marshak's solution). But more importantly, as discussed in the following paragraphs, it is a close approximation to the exact solution over a large parameter regime.

A measurement of the validity of the approximation is given by the residual

$$R(u) = n\varphi_T\varphi_T'' + (\varphi_T')^2 - K(\beta n\varphi_T - u\varphi_T').$$

This function is squared to make it positive definite and then integrated over the interval $0 \leqslant u \leqslant 1$. The resulting functional takes the form

$$L = \int_0^1 R^2\,\mathrm{d}u = L(a, n, \beta, K).$$

Minimising $L$ with respect to $K$ and solving for $K$ yields

$$K = \frac{-[2n(a-1)+a^2]a_0 + a(a-1)(2n+4)a_1 - (a-1)^2(2n+4)a_2}{-\beta n a_0 + a(\beta n - 1)a_1 + (a-1)(2-\beta n)a_2} \tag{16}$$

where

$$a_0 = \tfrac{1}{3}(a-4)(\beta n+1) \qquad a_1 = \tfrac{1}{6}(a-3)(\beta n+2) \qquad a_2 = \tfrac{1}{10}(a-\tfrac{8}{3})(\beta n+3).$$

Figures 1($a$) and ($b$) are plots of $K$ against $a$ for $n = 2.5$ and 5. The large closed points of figure 1($b$) represent the values of $K$ and $a$ obtained from the numerical integration of equation (14). Figure 2 is a plot of the trial function (15) for the two

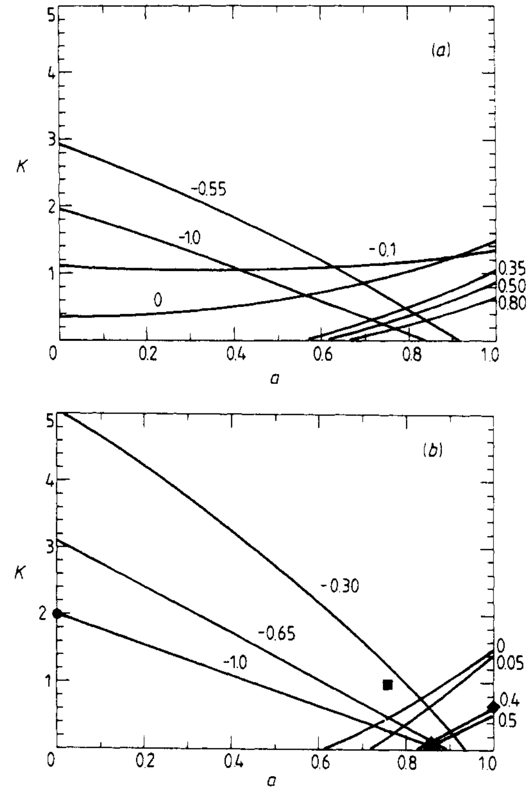


**Figure 1.** A plot of the eigenvalue $K$ against $a$ for the case of ($a$) electron thermal conduction, i.e. $n = 2.5$ and ($b$) radiation thermal conduction, i.e. $n = 5$. The parameter in both figures is $\beta$.

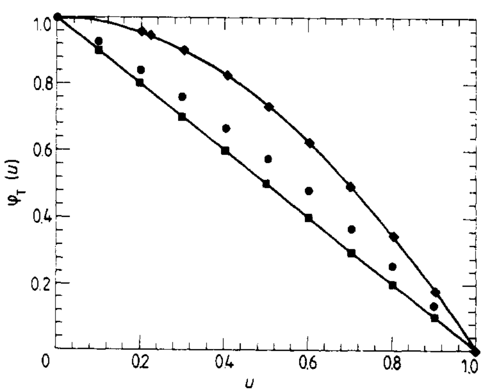


**Figure 2.** The full curves are plots of the trial function $\varphi_T(u)$ for $a = 0$ and 1. The points are the results of numerical integration of the exact differential equation (14) for parameter choices given as follows; ●, $K = 0.2$, $\beta = -1$, $a = 0.76$; ■ $K = 0.65$, $\beta = 0.4$, $a = 1.0$; ◆ $K = 2$, $\beta = -1$, $a = 0$.

extreme cases of figure 1(*b*), $a = 0$ and 1. Also shown are the numerical integrations (points) of these cases and an intermediate case, $a = 0.76$. It is clear from these two figures that the trial function is quite close to the exact solution over a reasonable parameter range.

Notice that the above results yield a value for $\xi_0$ and hence an expression for the heat front. Thus, the front of the heat wave is always located at

$$x_f = \xi_0 Z(t) L_0 \tag{17}$$

with

$$\xi_0 = (K(\beta, n)/n)^{1/2}. \tag{18}$$

From equations (2) and (3), the flux at the boundary is given by

$$S_0 = -\rho c_v A (T^n \nabla T)_{x=0} = -\frac{\rho c_v A T_0^{n+1} g^{n+1}}{Z L_0} \left.\frac{\mathrm{d}h}{\mathrm{d}\xi}\right|_{\xi=0}$$

or, using $g = Z^\beta$,

$$S_0 = -\rho c_v \nu L_0 T_0 z^{\beta(n+1)-1} \left.\frac{\mathrm{d}h}{\mathrm{d}\xi}\right|_{\xi=0}. \tag{19}$$

In terms of the trial function, the flux at the boundary is approximated by

$$S_0 = \rho c_v \nu L_0 T_0 a (nK)^{-1/2} Z^{\beta(n+1)-1}. \tag{20}$$

Notice here that $S_0$ scales as $Z^{\beta(n+1)-1}$ while, from equation (3), the temperature at the boundary $T(0, t)$ scales as $Z^\beta$.

Now that the temperature distribution in the heat wave has been determined, it is possible to compute the total energy in the thermal wave at time $t$. This is given by

$$E(t) = n_i c_v A_h \int_0^{x_f(t)} T_0 g(t) h(\xi)\, \mathrm{d}x \tag{21}$$

where $A_h$ is the surface area of the heated region and $n_i$ is the ion number density. Substituting $\varphi_T$ into equation (21) yields

$$E(t) = n_i c_v A_h T_0 L_0 (K/n)^{1/2} Z^{\beta+1} I(a). \tag{22}$$

The integral

$$I(a) = \int_0^1 \varphi_T^{1/n}(u)\, \mathrm{d}u \tag{23}$$

has the value

$$I(a) = [(2-a)/(1-a)][(2-a)^2/(1-a)]^{1/n} B(1+(1/n), 1+(1/n), (1-a)/(2-a)) \tag{24}$$

where $B(x, y, z)$ is the incomplete beta function (Abramowitz and Stegun 1970). We note here that according to equation (22) the energy is constant when $\beta = -1$. This corresponds to the first of the four special cases described below.

## 4. Special cases

There are four cases of special interest in the nonlinear thermal conduction problem. They correspond to the physical problems that have (i) energy released at the boundary (Zeldovich's solution), (ii) constant temperature at the boundary, (iii) constant flux at the boundary, and (iv) exponential temperature dependence at the boundary (Marshak 1958). We will describe these four cases separately.

### *4.1. Energy released at the boundary*

This case corresponds to the physical problem of rapid (compared with the heat wave motion) heating of a boundary layer of material which drives a heat wave with the average temperature dropping as more matter is heated. This problem has been examined extensively by Zeldovich and Raizer (1967). Their solution, however, has finite energy released in zero initial material thickness, giving a somewhat unrealistic infinite temperature at the boundary at $t = 0$. In our formulation, the energy is released in a finite initial thickness of dimension $x_f(0) = \xi_0 L_0$. In this case, we have to choose $\beta = -1$ and $a = 0$. Then from equation (16) we find $K = 2$, and the trial function $\varphi_T = (1 - u^2)$. Zeldovich has shown this function to be an exact solution, with the thermal wave given by

$$T(x, t) = (T_0/Z(t))(1 - x^2/x_f^2(t))^{1/n} \tag{25}$$

where $x_f(t) = (2/n)^{1/2} L_0 Z(t)$ and $Z(t) = [1 + (2+n)\nu t]^{1/(2+n)}$. Here the constant, $\nu$, is not the heating rate but rather can be related to the total energy deposited in the heated layer at $t = 0$. We have $\nu = AT_0 n/L_0^2$ and $T_0$ is determined from the energy integral. Using equations (22) and (24) with $a = 0$, we have

$$E_0 = \tfrac{1}{2} n_i c_v A_h T_0 L_0 (2/n)^{1/2} \Gamma(\tfrac{1}{2}) \Gamma(1 + (1/n)) / \Gamma(\tfrac{3}{2} + 1/n) \tag{26}$$

where $A_h$ is the surface area of the heated foil and where identities from Abramowitz and Siegun (1970, ch 6) are used to rewrite the incomplete beta function in terms of the gamma function.

### *4.2. Constant temperature at the boundary*

This case corresponds to the parameter choice $\beta = 0$. The front of the heat wave moves according to

$$x_f(t) = (K/n)^{1/2} (AT_0^n/\nu)^{1/2} (1 + 2\nu t)^{1/2} \tag{27}$$

and the flux at the boundary is given by

$$S_0(t) = \rho c_v \nu L_0 T_0 (nK)^{-1/2} a (1 + 2\nu t)^{1/2}. \tag{28}$$

Here the time constant, $\nu$, is related to the rate of change of the flux at the boundary (decreasing) required to maintain a constant boundary temperature. In both of the above expressions we have chosen to display the dependence of the heat wave front on the initial parameters, $T_0$, $\nu$, and $S_0(0)$. We note that the energy delivered to the material (see equation (22)) increases as

$$E(t) \sim (1 + 2\nu t)^{1/2}. \tag{29}$$

*4.3. Constant flux at the boundary*

This case corresponds to $\beta = (n+1)^{-1}$ as is seen from equation (20). The heat front moves according to

$$x_f(t) = (K/n)^{1/2} L_0 \{1 + [(n+2)/(n+1)]\nu t\}^{(n+1)/(n+2)} \tag{30}$$

where $n < 4$. The parameter $\nu$ is simply determined in terms of the boundary flux and the initial boundary values $L_0$ and $T_0$. From equation (20), we find that

$$\nu = [(nK)^{1/2}/(\rho c_v L_0 T_0 a)] S_0 \tag{31}$$

and, of course, the temperature at the boundary increases as

$$T(0, t) = T_0 Z^{1/(n+1)} = T_0 \{1 + [(n + 2/(n+1)]\nu t\}^{1/(n+2)}. \tag{32}$$

*4.4. Exponential temperature dependence at the boundary*

This case corresponds to Marshak's (1958) problem, where the boundary temperature starts at $T_0$ and increases exponentially with time and with a heating rate of $\nu$. In our formulation, this case has $\beta = 2/n$ and as can be seen from figures 1($a$) and ($b$), we have $a \approx 1$. The heat front moves according to

$$x_f(t) = (K/n)^{1/2} L_0 \, e^{\nu t} \tag{33}$$

and the thermal wave profile is given approximately by

$$T(x, t) = T_0 \exp(2\nu t/n)(1 - x/x_f)^{1/n} \tag{34}$$

and the total energy obtained from equation (22) increases as

$$E(t) \sim \exp[(1 + 2/n)\nu t]. \tag{35}$$

We have described a class of self-similar solutions to a variety of nonlinear thermal conduction problems. Our formulation displays the similarity between the previous work of Zeldovich and Marshak and extends the class of self-similar solutions as well. In addition, we have derived the scaling relations between the various physical parameters and introduced physically reasonable boundary conditions for the start-up of the nonlinear heat wave.

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## References

Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions, NBS Appl. Maths Series #55* (New York: Dover)

Becker M 1964 *The Principles and Applications of Variational Methods* (Cambridge, Mass.: MIT)

Gilding B H and Peletier L A 1976 *J. Math. Anal. Appl.* **55** 351–64
Grundy R E 1979 *Appl. Math.* **37** 259–80
Marshak R E 1958 *Phys. Fluids* **1** 24–9
Zeldovich Ya and Raizer Yu A 1967 *Physics of Shock Waves and High Temperature Hydrodynamic Phenomena* (New York: Academic)